Chapter: Posets & Lattices

Partially Ordered sets (Posets)

Partial Order — A relation R on set A is called a partial order if R is reflexive, antisymmetric and transitive.

Partially Ordered Set (Poset) — The set A together with the partial order R is called a partially ordered set, or poset.
It is denoted as (A,R).

Example 1: — Identify whether each of the following relations is a partial order or not.

- $\subseteq$ on set A which is a collection of subsets of a set S.
- $\leq$ on set Z^+
- divisibility on Z^+
- $<$ on Z^+

Example 2: — On the set A ={a,b,c}, find all partial orders $\leq$ in which $a\leq b$.

Try it yourself:
Problem 1: — Determine whether the relation R is a partial order on set A.

(a) A=Z. and aRb if & only if a=2b.
(b) A=Z. & aRb if & only if $b^2|a$.
(c) A=Z. & aRb if & only if $a=b^k$ for some $k\epsilon Z^+$.
(d) A=R (real number set) & aRb if & only if $a\leq b$.

Problem 2: — What can you say about the relation R on a set A if R is a partial order & an equivalence relation?

Note: If R is a partial order then R^{-1} too is a partial order.
Thus (A,R^{-1}) is called the dual of the poset (A,R), & the partial order R^{-1} is called the dual of the partial order R.

Note: Further in this chapter $\leq$ shall be used to signify any relation R which is a partial order. Do not confuse yourself in treating this as the actual relation $\leq$ on Z.

Comparable — If $(A,\leq)$ is a poset, the elements a & b of A are said to be comparable if $a\leq b$ or $b\leq a$

Note: In a partially ordered set every pair of elements need not be comparable.

Linearly ordered set
Linear order
Chain — If every pair of elements in a poset A i.e. (A,R) is comparable, we say that A is a linearly ordered set, & the partial order is called a linear order. We also say that A is a chain.
e.g. The poset $(\leq,Z^+)$ is linearly ordered.

Theorem 1: — If $(A,\leq)$ & $(B,\leq)$ are posets, then $(AXB,\leq)$ is a poset, with partial order $\leq$ defined as
$(a,b)\leq(a',b')$ if $a\leq a'$ in A & $b\leq b'$ in B

Product Partial Order — The partial order $\leq$ defined on the Cartesian product AXB as above is called the product partial order.

Try it yourself:
Problem 3: — Determine whether the relation R is a linear order on the set A.

(a) A=R & aRb if & only if $a\leq b$.
(b) A=R & aRb if & only if $a\geq b$.
(c) A=P(S), where S is a set. The relation R is set inclusion.
(d) A=RXR & (a,b)R(a',b') if & only if $a\leq a'$ & $b\leq b'$, where $\leq$ is the usual partial order on R.

Lexicographic ordering — Lexicographic ordering or dictionary ordering:
If $(A,\leq)$ is a poset then $a<b$ is defined if $a\leq b$, but $a\neq b$.
Suppose $(A,\leq)$ & $(B,\leq)$ are posets, then the product partial order on AXB can be defined as in Theorem 1.
Another useful partial order on AXB, denoted by $\prec$, is defined as follows:
$(a,b) \prec (a',b')$ if $a<a'$ or $a=a'$ & $b\leq b'$. This is called lexicographic or dictionary order. The ordering of the elements in the first coordinate dominates, except in case of ties, when attention passes on to the second coordinate.
If $(A,\leq)$ & $(B,\leq)$ are linearly ordered sets, then the lexicographic order $\prec$ on AXB is also a linear order.
Lexicographic ordering is easily extended to Cartesian products $A_1XA_2X\ldots XA_n$ as follows:
$(a_1,a_2,\ldots\ldots,a_n) \prec (a_1',a_2',\ldots\ldots,a_n')$ if & only if
$a_1 < a_1'$ or
$a_1=a_1'$ & $a_2<a_2'$ or
$a_1=a_1'$ & $a_2=a_2'$ & $a_3<a_3'$ or
……………………
$a_1=a_1'$, $a_2=a_2'$, ………., $a_{n-1} = a'_{n-1}$ & $a_n\leq a_n'$
Thus the first coordinate dominates except during equality, in which case we consider the second coordinate. If equality holds again then we pass on to the next coordinate, & so on.

Example 3: — Let S={a,b,c,………,z}, linearly ordered in the usual way ($a\leq b, b\leq c,\ldots, y\leq z$), then what is S^n & how is lexicographic ordering done on S^n. What is S* & how is lexicographic ordering done on S*.

Try it yourself:
Problem 4: — Let $A=Z^+XZ^+$ have lexicographic order. Mark each of the following as true or false.

(a) $(15,92) \prec (12,3)$ (b) $(4,8) \prec (4,6)$
(c) $(3,6) \prec (3,24)$ (d) $(2,12) \prec (5,3)$

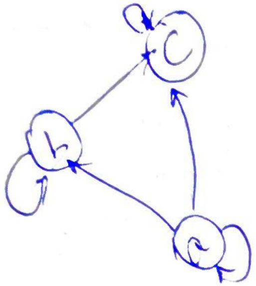
Problem 5: — Let A={□, A,B,C,E,O,M,P,S} have the usual alphabetical order, where □ represents a blank character & $□\leq x$ for all $x\epsilon A$. Arrange the following in lexicographic order.

Theorem 2: — The digraph of a partial order has no cycles of length greater than 1.

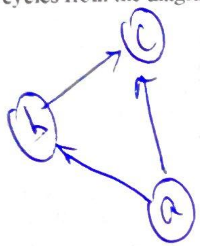
Hasse Diagrams

Construction of Hasse diagram

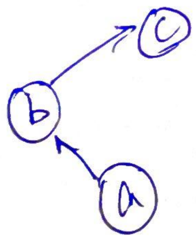
Step1: Draw the diagraph of the partial order. (It has cycles of length 1).



Step 2: Delete all the cycles from the diagraph.



Step 3: Delete all the edges that imply transitive property. i.e if a≤b & b≤c then delete a≤c.



Step 4: Draw the digraph with all edges pointing upwards, so the arrows are omitted from the diagraph.

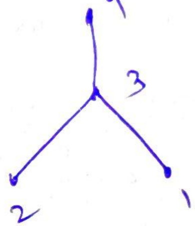
Step 5: Replace all the circles representing vertices by dots.

Example 4: Let A={1,2,3,4,12}. Consider the partial order of divisibility on A & draw the Hasse diagram for the same.

Example 5: Let S={a,b,c} & A=P(S). Draw the Hasse diagram of the poset A with the partial order ⊆. Is this a linearly ordered set?

Note: If (A,≤) is a poset, then (A,≥) is the dual poset. The Hasse diagram of (A,≥) is the Hasse diagram of (A,≤) turned upside down.

Example 6: Describe the ordered pairs in the relation determined by the hasses diagram on the set A.
A={1,2,3,4}



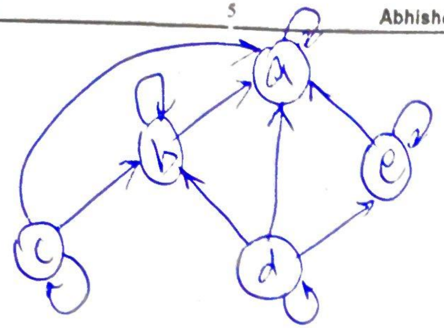
Try it yourself:

Problem 6: A={1,2,3,4}
R={(1,1),(1,2),(2,2),(2,4),(1,3),(3,3),(3,4),(1,4),(4,4)}
Draw its Hasse diagram.

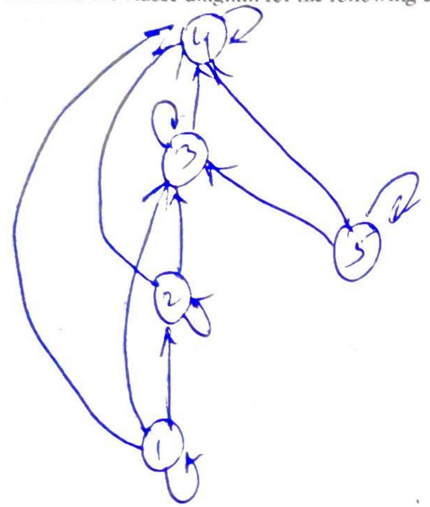
Problem 7: A={a,b,c,d,e}
R={(a,a),(b,b),(c,c),(a,c),(c,d),(c,e),(a,d),(d,d),(a,e),(b,c),(b,d),(b,e),(e,e)}
Draw its Hasse diagram.

Problem 8: Determine the ordered pairs of the relation whose Hasse diagram is:

Problem 9: Determine the Hasse diagram for the following diagraph.



Problem 10: Determine the Hasse diagram for the following diagraph:



Problem 11: Determine the Hasse diagram of the relation on A={1,2,3,4,5} whose matrix is shown:

(a)

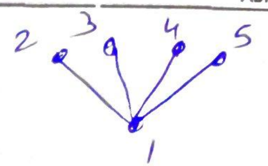
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b)

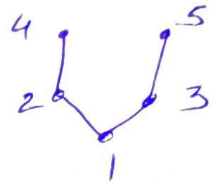
$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 12: Determine the matrix of the partial order whose Hasse diagram is given as

(a)



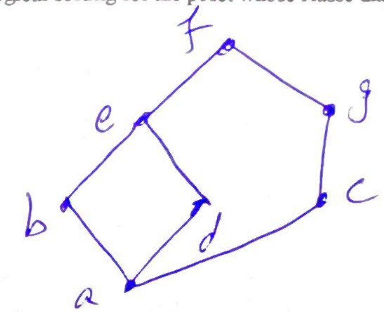
(b)



Topological Sorting

Topological Sorting If A is poset with partial order ≤ & we want to find a linear order ≺ for the set A that will merely be an extension of the given partial order in the sense that if a≤b, the a ≺ b. The process of creating a linear order such as ≺ is called topological sorting.

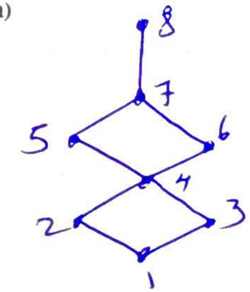
Example 7: Give a topological sorting for the poset whose Hasse diagram is:



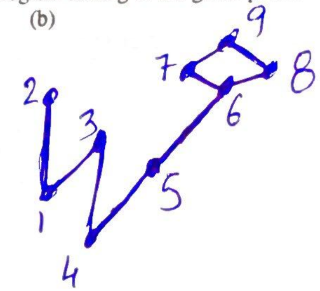
Try it yourself:

Problem 13: Draw the Hasse diagram of a topological sorting of the given poset:

(a)



(b)



Example 8: Show that if R is a linear order on the set A, then R^{-1} is also a linear order on A.

Example 9: A relation R on set A is called a quasiorder if it is transitive & irreflexive. Let A=P(S) be the power set of S, & consider U & T ε A then URT if and only if U T (proper containment). Show that R is a quasi – order.

Try it yourself:

Problem 14: Let A={x|x is a real number & $-5 \le x \le 20$}. Show that the usual relation < is a quasiorder on A.

Problem 15: If R is a quasiorder on A show that R^{-1} is a quasiorder.

Problem 16: Let B ={2,3,6,9,12,18,24} & let A=BXB. Define the following relation on A: (a,b) ≺ (a',b') if & only if a|a' & b≤b', where ≤is the usual partial order. Show that ≺ is a partial order.

Isomorphism

Isomorphism & Isomorphic posets Let (A,≤) & (A',≤') be posets & let f :A→A' be a one – to – one correspondence between A & A'. The function f is called an **isomorphism** from (A,≤) to (A',≤') if, for any a & b in A,
a≤b if & only if f(a)≤'f(b).
If f :A→A is an isomorphism, we say that (A,≤) & (A',≤') are **isomorphic posets.**

Example 10: Let A be the set Z^+ of positive integers, and let ≤ be the usual partial order on A. Let A' be the set of positive even integers, and let ≤' be the usual partial order on A'. Is the function f : A→A' given by f(a)=2a an isomorphism from (A,≤) to (A',≤).

Note: Two finite isomorphic posets must have the same Hasse diagram.

Example 11: Let A={1,2,3,6} & let ≤ be the relation | (divides). Let A' =P({a,b}) & let ≤' be set containment, ⊆. If f: A→A' then is f an isomorphism.

Try it yourself:

Problem 17: Let A ={1,2,3,5,6,10,15,30} & consider the partial order ≤ of divisibility on A. That is, define a≤b to mean that a|b. Let A'=P(S), where S={e,f,g}, be the poset with partial order ⊆. Show that (A,≤) & (A', ⊆) are isomorphic.

Problem 18: Let A={1,2,4,8} & let ≤ be the partial order of divisibility on A. Let A' = {0,1,2,3} & let ≤' be the usual relation "less than or equal to" on integers. Show that (A,≤) & (A',≤') are isomorphic posets.

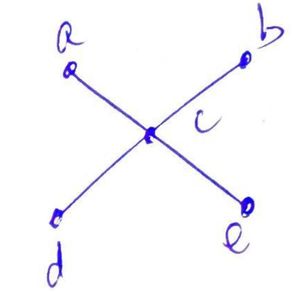
Extremal Elements of POSETS

Maximal element If ≤ is the usual partial order then a ε A is called a **maximal element** of A if there is no element c in A such that a<c.

Minimal element If ≤ is the usual partial order then b ε A is called a **minimal element** of A if there is no element c in A such that c<b.

Note: If (A,≤) is a poset, then its dual poset (A,≥) is obtained by inverting the Hasse diagram of A. Thus, a ε A is a maximal element of (A,≥) if & only if a is a minimal elelment of (A,≤). Also, a is a minimal element of (A,≥) if & only if it is a maximal element of (A,≤).

Example 12: Find the maximal & minimal elements of poset A whose Hasse diagram is given as follows:



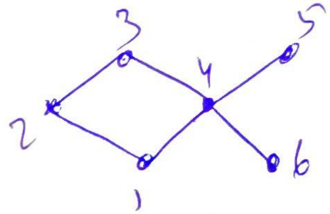
Example 13: Let A be a poset of nonnegative real numbers with the usual partial order ≤. Find its minimal & maximal elements.

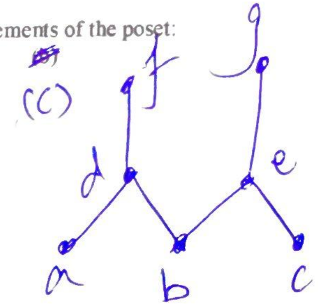
Theorem: Let A be a finite nonempty poset with the partial order ≤. Then A has at least one maximal element & at least one minimal element.

Try it yourself:

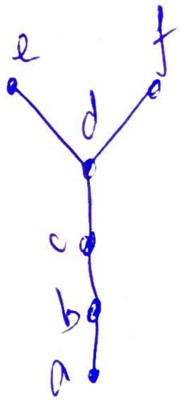
Problem 19: Determine all maximal & minimal elements of the poset:

(a)

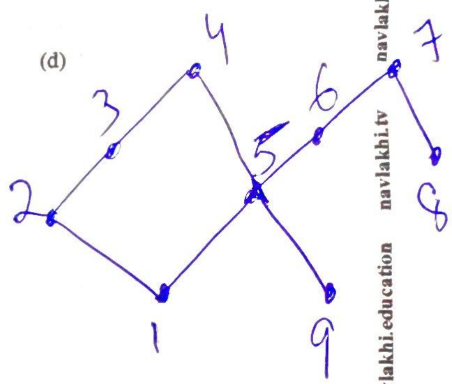




(b)



(d)



Problem 20: Find the maximal & minimal elements of the following poset:

(a) A=R with the usual partial order ≤.
(b) A={x|x is a real number & $0 \le x < 1$} with the usual partial order ≤.
(c) A={x|x is a real number & $0 < x \le 1$} with the usual partial order ≤.
(d) A={2,3,4,6,8,24,48} with the partial order of divisibility.

Greatest & Least element

An element a ε A is called a **greatest element** of A if x ≤ a for all x ε A.
An element a ε A is called a **least element** of A if a ≤ x for all x ε A.

Example 14:

Find the greatest & least element of the following posets:
(a) A, the set of non negative real numbers with the usual partial order ≤.
(b) A=P(S) where S={a,b,c}.
(c) Z, with the usual partial order of ≤.
(d) A={x|x is a real number & 0≤ x ≤ 1} with the usual partial order ≤.
(e) A={ 2,3,4,6,12,18,24,36} with the partial order of divisibility.

Theorem:

A poset has at most one greatest element & at most one least element.

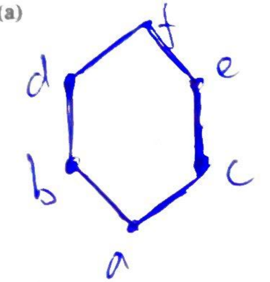
Note:

The greatest element of a poset, if it exists, is denoted by I & is often called the unit element.
The least element of a poset, if it exists, is denoted by 0 & is often called the zero element.

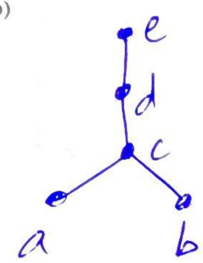
Example 15:

Determine the greatest & least elements, if they exist, of the poset:

(a)



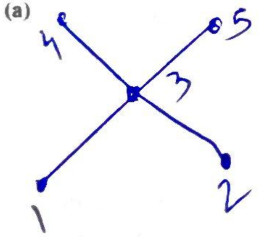
(b)



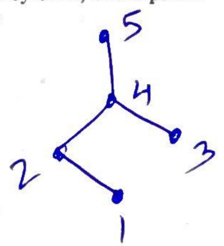
Try it yourself:
Problem 21:

Determine the greatest & least elements, if they exist, of the poset:

(a)



(b)



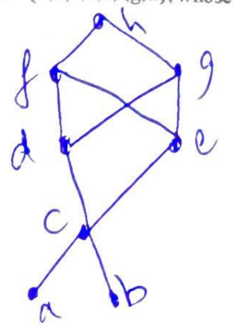
(c) A={x|x is a real number & 0 < x < 1}with the usual partial order ≤.
(d) A={2,4,6,8,12,18,24,36,72} with the partial order of divisibility.

Upper bound & Lower bound

Consider a poset A & a subset B of A. An element a ε A is called an **upper bound** of B if b ≤ a for all b ε B.
Consider a poset A & a subset B of A. An element a ε A is called an **lower bound** of B if a ≤ b for all b ε B.

Example 16:

Consider a poset A={a,b,c,d,e,f,g,h}, whose Hasse diagram is shown below:



Find all upper bounds & lower bounds of the following subsets:
B_1 = {a,b} B_2 ={c,d,e}

Least Upper Bound (LUB) & Greatest Lower Bound (GLB)

An element a ε A is called a **least upper bound** (LUB) of B if a is an upper bound of B & a ≤ a', whenever a' is an upper bound of B.
An element a ε A is called a **greatest lower bound** (GLB) of B if a is a lower bound of B & a' ≤ a, whenever a' is a lower bound of B.

Note:

The upper bound of (A,≤) corresponds to the lower bound of its dual (A,≥).
The lower bound of (A,≤) corresponds to the upper bound of its dual (A,≥).
The least upper bound of (A,≤) corresponds to the greatest lower bound of its dual (A,≥).
The greatest lower bound of (A,≤) corresponds to the least upper bound of its dual (A,≥).

Example 17:

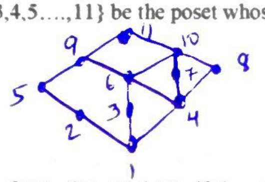
Find all LUB & GLB of B_1 & B_2 in example 16.

Theorem:

Let (A,≤) be a poset. Then a subset B of A has at most one LUB & at most one GLB.

Example 18:

Let A={1,2,3,4,5....,11} be the poset whose Hasse diagram is:



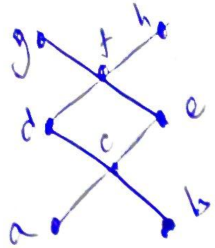
Find LUB & GLB of B={6,7,10}, if they exist.

Try it yourself:
Problem 22:

Find (a) all upper bounds of B (b) all lower bounds of B (c) the LUB of B (d) the GLB of B.

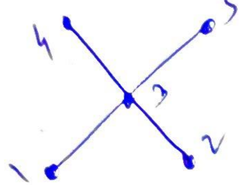
(i)

PTO



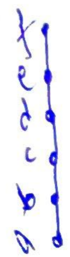
B={c,d,e}

(ii)



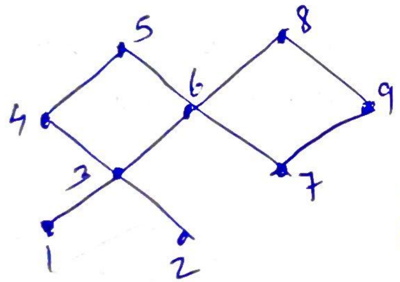
B={1,2,3,4,5}

(iii)



B={b,c,d}

(iv)



B={3,4,6}
B={4,6,7}

Problem 22: Find (a) all upper bounds of B (b) all lower bounds of B (c) the LUB of B (d) the GLB of B.

(i) (A,≤) is the poset of figure (i) above & B={b,g,h}.
(ii) (A,≤) is a poset of figure (iv) above & B={4,6,9}
(iii) (A,≤) is a poset of figure (iv) above & B={3,4,8}
(iv) A=R & ≤ denotes the usual partial order. B={x|x is a real number & 1<x<2}
(v) A=R & ≤ denotes the usual partial order. B={x|x is a real number & 1≤x<2}
(vi) A=P({a,b,c}) & ≤ denotes the partial order of containment; B=P({a,b})

(vii) A={2,3,4,6,8,12,24,48} & ≤ denotes the partial order of divisibility; b={4,6,12}.

LATTICES:

What is a lattice? A lattice is a poset (L,≤) in which every subset {a,b} has a least upper bound & a greatest lower bound.

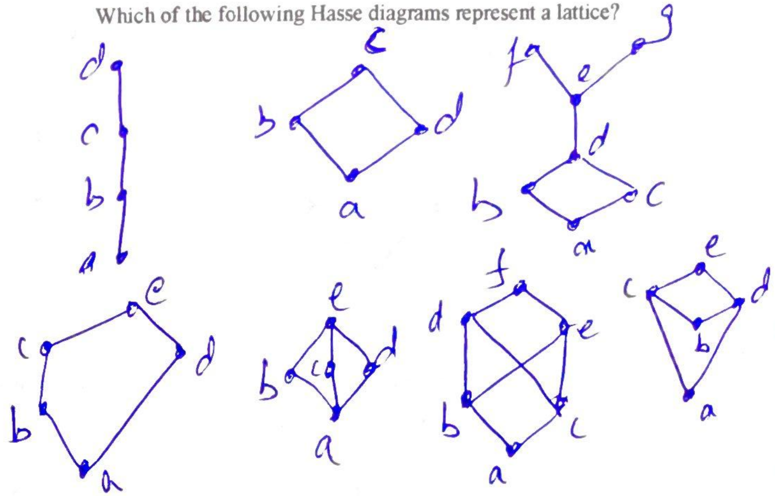
Notation LUB({a,b}) is given by aVb called join of a & b.
GLB({a,b}) is given by a b called meet of a & b

Example 19: Let S be a set & L=P(S) & let ⊆ be the partial order on L. Show that L is a lattice.

Example 20: Consider the poset (Z^+,≤) where a≤b if & only if a|b. Is the poset a lattice.

Example 21: Let n be a positive integer & let D_n be the set of all positive divisors on n. Is D_n a lattice? Draw the Hasse diagram for D_{20} & D_{30}.

Example 22: Which of the following Hasse diagrams represent a lattice?

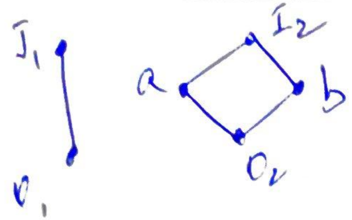


Note: If (L,≤) is a poset that is a lattice, then (L,≥) is a dual poset which too is a lattice. The LUB of a & b in (L,≤) is the GLB of a & b in (L,≥) & vice versa.

Theorem 3: If (L_1,≤) & (L_2,≤) are lattices, then (L,≤) is a lattice, where L=L_1 X L_2, & the partial order ≤ of L is the product partial order.

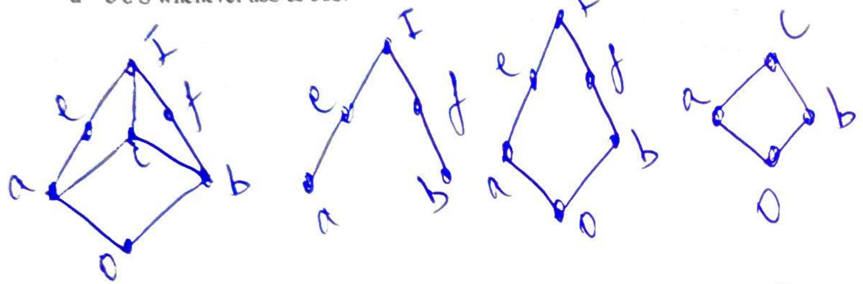
Note: $(a_1,b_1)V(a_2,b_2) = (a_1 V_1 a_2, b_1 V_2 b_2)$

Example 23: Find L = L_1 X L_2.



Sublattice A nonempty subset S of lattice (L,≤) is called a sublattice of L if aVb ε S & a b ε S whenever aεS & bεS.

Example 24:



Which of the following is a sublattice of the above lattice.

Properties of Lattices:

Note:

Theorem 4:

(i) aVb = b if & only if a≤b

(ii) $a \wedge b = a$ if & only if $a \leq b$

(iii) $a \wedge b = a$ —"— $a \vee b = b$

Example 25: Is a linearly ordered poset a lattice.

Theorem 5: Idempotent Properties

$a \vee a = a$
$a \wedge a = a$

Commutative Properties

$a \vee b = b \vee a$
$a \wedge b = b \wedge a$

Associative Properties

$a \vee (b \vee c) = (a \vee b) \vee c$
$a \wedge (b \wedge c) = (a \wedge b) \wedge c$

Absorption Properties

$a \vee (a \wedge b) = a$
$a \wedge (a \vee b) = a$

Theorem 6: if $a \leq b$ then $a \vee c \leq b \vee c$
$a \wedge c \leq b \wedge c$

$a \leq c$ & $b \leq c$ if & only if $a \vee b \leq c$
$c \leq a$ & $c \leq b$ if & only if $c \leq a \wedge b$
if $a \leq b$ & $c \leq d$ then $a \vee c \leq b \vee d$
$a \wedge c \leq b \wedge d$

Types of Lattices:

Bounded Lattice: A lattice L is said to be bounded if it has a greatest element I & has a least element 0.

Example 26: Is the lattice Z under the usual partial order ≤ bounded?

Example 27: Consider the lattice P(S) under the partial order of ⊆. Is the lattice bounded?

Theorem 7: Let L= $\{a_1, a_2, \ldots, a_n\}$ be a finite lattice. Then L is bounded.

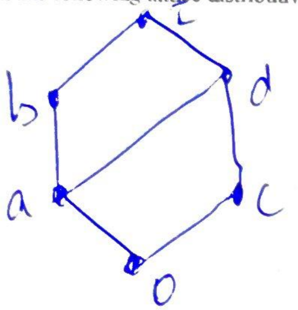
Distributive lattice: A lattice L is called distributive if for any elements a,b & c in L we have the following distributive properties.

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

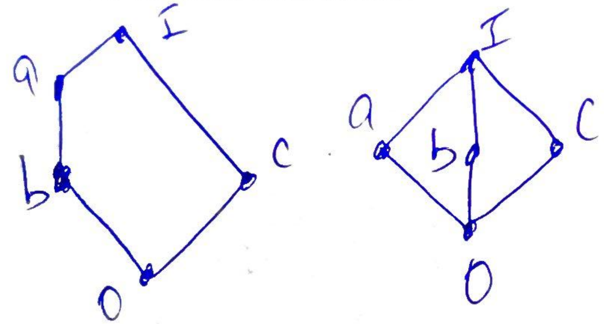
If L is not distributive we say that L is nondistributive.

Example 28: Is the lattice of Example 27 distributive?

Example 29: Is the following lattice distributive?



Example 30: Show that the lattice given below is nondistributive.



Theorem 8: **A lattice L is nondistributive if & only if it contains a sublattice that is isomorphic to one of the two lattices.**

Complement of an element: Let L be a bounded lattice with greatest element I & least element 0, & a∈L. An element a'∈L is called a complement of a if

$a \vee a' = I$ & $a \wedge a' = 0$

Note: $0' = I$ & $I' = 0$

Example 31: For lattice of Example 30 find the complement of each element

Note: An element a in a lattice need not have a complement & it may have more than one complement.

Theorem 9: Let L be a bounded distributive lattice. If a complement exists, it is unique.

Complemented Lattice: A lattice L is called complemented if it is bounded & if every element in L has a complement.

Example 32: Is the lattice of Example 27 complemented?

Example 33: Are the lattices of Example 30 complemented?

Isomorphic Lattices:

Isomorphic Lattices: If $f: L_1 \to L_2$ is an isomorphism from the poset $(L1, \leq 1)$ to the poset $(L2, \leq 2)$, then if a & b are elements of L1, then

If two lattices are isomorphic then we say they are isomorphic lattices.

Example 34: Let S={a,b,c} & L=P(S). Prove that $(L, \subseteq)$ is isomorphic to D_{42}.